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Elasticity recovery correspondence principles for physically nonlinear viscoelastic problems for a class of materials

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Abstract

Two new correspondence principles, which are called elasticity recovery correspondence principles, for problems involving a class of nonlinear (or linear) viscoelastic materials in one-dimensional case are proposed in this paper. By means of these principles, solutions to nonlinear viscoelastic problems can be obtained as long as the solutions to the corresponding nonlinear elastic problems exist. The idea of these principles is entirely different from the traditional one. Not the similarity between the elastic constitutive relation and the viscoelastic relation is utilized. Rather, the recoverability from the nonlinear viscoelastic response to the nonlinear instantaneous elastic response is utilized. It is shown by experiments for modified polypropylene that these principles are applicable for such a class of materials. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Elasticity recovery; Correspondence principle; Nonlinear viscoelasticity; Constitutive relation

1. Introduction

In linear viscoelasticity, the solutions of viscoelastic problems have been greatly simplified by using the correspondence principle (see e.g. Christensen, 1982; Zhang, 1994). Attempts have been made in finding the correspondence principles for solving nonlinear viscoelastic problems by some authors, such as Schapery (1982, 1984), Hu and Tong (1991), etc. Simplified nonlinear viscoelastic constitutive relations were given by Rabotnov (1980) and a theory of elastic–plastic heredity was discussed based on the concept of the modified stress. When the strain $\varepsilon(t)$ is prescribed, Schapery (1982) defined a pseudostrain ε^{o}, which is related to the current strain ε by taking a hereditary integral. He expected that the relation between the stress σ and the pseudostrain ε^{o} could behave like that of a nonlinear elastic material. However, in his example (Schapery, 1982) for a nonlinear viscoelastic material subjected to ten equal-amplitude cyclic strain, he also found himself that the pseudostrain and stress could not return to the origin simultaneously and the loading curves and the unloading curves could not coincide completely with each other. Hu and Tong adopted the nonlinear viscoelastic constitutive equations in the form of n-multiple integral:

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$$\sigma_{ij}(t) = \int_0^{\infty} a_1(t-\tau_1)_{ijkl}\dot{\varepsilon}_{kl}(\tau_1)\,\mathrm{d}\tau_1 + \int_0^{\infty}\int_0^{\infty} a_2(t-\tau_1, t-\tau_2)_{ijklmn}\dot{\varepsilon}_{kl}(\tau_1)\,\dot{\varepsilon}_{mn}(\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 + \cdots .$$

Using the n-multiple Laplace transformation, they proposed a correspondence principle for nonlinear viscoelastic materials. Since too much material kernel functions and parameters are included in the constitutive relations, it is not convenient for practical application. The considerable difficulties involved in finding solutions of nonlinear viscoelastic problems in the general cases compel us to use the simplified nonlinear viscoelastic constitutive relations and to focus our attention in physical nonlinearity. In this paper, attempt is made in finding corresponding principles in the range of infinitesimal deformation based on the simplified nonlinear viscoelastic constitutive relations and only the one-dimensional case will be discussed. The idea in finding the correspondence principles is entirely different from that in linear viscoelastic theory. Not the similarity between the nonlinear viscoelastic constitutive relation and the nonlinear elastic constitutive relation is utilized. Rather, the recoverability of the nonlinear current viscoelastic response to the instantaneous elastic response is utilized.

The paper is organized as follows: Section 2 discusses the correspondence between the nonlinear viscoelastic and nonlinear elastic constitutive relations. Section 3 is devoted to the nonlinear elastic–viscoelastic correspondence principles, which we shall call the elasticity recovery correspondence principles. A comparison between theory and experiments is given in Section 4. Section 5 closes the paper with some concluding remarks.

2. Correspondence between the nonlinear viscoelastic and nonlinear elastic constitutive relations

2.1. Relations between the current stress (or strain) and the recovered elastic stress (or strain)

Let t be the time, x, $\sigma(x,t)$, $\varepsilon(x,t)$ and $u(x,t)$ be the position, the current stress, the current strain and the current displacement in one-dimensional case, respectively. Assume that the viscoelastic material possesses instantaneous elastic response. Let $\sigma^{\mathrm{e}}(x,t)$, $\varepsilon^{\mathrm{e}}(x,t)$ and $u^{\mathrm{e}}(x,t)$ be the instantaneous elastic stress, the instantaneous elastic strain and the instantaneous elastic displacement, respectively. A material possessing elastic property is defined as: the material obeys the first and the second laws of thermodynamics, and the loading process is reversible. This definition requires that the loading curves and the unloading curves must fall in the same curve, and the stress and the strain must return to the origin simultaneously. It follows that (see e.g. Fung, 1965) there exists a strain energy function $W = W(\varepsilon^{\mathrm{e}}, x, t)$ with the property that

$$\sigma^{\mathrm{e}} = \partial W/\partial\varepsilon^{\mathrm{e}} \tag{1}$$

and a complementary strain energy function $W_{\mathrm{c}} = -W + \sigma^{\mathrm{e}}\varepsilon^{\mathrm{e}} = W_{\mathrm{c}}(\sigma^{\mathrm{e}}, x, t)$ with the property that

$$\varepsilon^{\mathrm{e}} = \partial W_{\mathrm{c}}/\partial\sigma^{\mathrm{e}}. \tag{2}$$

Eqs. (1) and (2) define the nonlinear elastic constitutive relations. To obtain the correspondence principles, let us at first search the relations between the current stress σ (or strain ε) and the instantaneous elastic stress σ^{e} (or strain ε^{e}). Generally, such relations are very complicated. It is necessary to simplify the nonlinear viscoelastic constitutive relations. The Volterra–Fréchet relation in the one-dimensional case is written as

$$\varepsilon(t) = \int_{-\infty}^{t} D_1(t-\tau_1)\,\mathrm{d}\sigma(\tau_1) + \int_{-\infty}^{t}\int_{-\infty}^{t} D_2(t-\tau_1, t-\tau_2)\,\mathrm{d}\sigma(\tau_1)\,\mathrm{d}\sigma(\tau_2) + \cdots \tag{3}$$

or its inverse relation

$$\sigma(t) = \int_{-\infty}^{t} E_1(t-\tau_1)\,\mathrm{d}\varepsilon(\tau_1) + \int_{-\infty}^{t}\int_{-\infty}^{t} E_2(t-\tau_1, t-\tau_2)\,\mathrm{d}\varepsilon(\tau_1)\,\mathrm{d}\varepsilon(\tau_2) + \cdots. \tag{4}$$

To fit the experimental data satisfactorily, more than three kernels D_i or E_i are needed, and the calculation of the kernels E_i from D_i, or conversely, involves great difficulties. Therefore, Eqs. (3) and (4) are not convenient for practical application. Following Rabotnov (1980), we suppose that the kernel D_i is the product of like functions of i different arguments

$$D_i(t-\tau_1, t-\tau_2, \ldots, t-\tau_i) = a_i \prod_{k=1}^{i} D_\sigma^{\mathrm{r}}(t-\tau_k), \tag{5}$$

where nondimensional quantity $D_\sigma^{\mathrm{r}}(t) = D_\sigma(t)/D_\sigma$ is referred to as the relative creep compliance. The subscript "σ" of D_σ^{r} in the σ-relation (6) is used to distinguish $D_\varepsilon^{\mathrm{r}}$ in the ε-relation (20b). $D_\sigma(t)$ is referred to as the creep compliance and $D_\sigma = D_\sigma(0)$ is the instantaneous elastic compliance. Substituting Eq. (5) into Eq. (3) and defining $\sigma^{\mathrm{e}}(t)$ as the following Stieltjes convolution

$$\sigma^{\mathrm{e}}(t) = D_\sigma^{\mathrm{r}} * \mathrm{d}\sigma = \int_{-\infty}^{t} D_\sigma^{\mathrm{r}}(t-\tau)\,\mathrm{d}\sigma(\tau), \tag{6}$$

we can rewrite Eq. (3) as

$$\varepsilon(t) = a_1\sigma^{\mathrm{e}}(t) + a_2[\sigma^{\mathrm{e}}(t)]^2 + a_3[\sigma^{\mathrm{e}}(t)]^3 + \cdots. \tag{7}$$

Series (7) defines $\varepsilon(t)$ as a function of $\sigma^{\mathrm{e}}(t)$. The inversion of this function is

$$\sigma^{\mathrm{e}}(t) = \varphi[\varepsilon(t)], \tag{8}$$

where $\varepsilon(t)$ is understood as an prescribed input quantity, so that the known strain itself is the instantaneous elastic strain, i.e. $\varepsilon(t) = \varepsilon^{\mathrm{e}}(t)$. This corresponds that $E_\varepsilon^{\mathrm{r}}(t) = H(t)$ in Eq. (11) (see Eq. (17)), where $H(t)$ is the Heaviside unit-step function. Eq. (8) defines an instantaneous elastic stress σ^{e} as a single-valued continuous function of ε^{e}, which defines the instantaneous elastic stress–strain relation. If the stress $\sigma^{\mathrm{e}}(t)$ is calculated from the current stress $\sigma(t)$ by the hereditary integral (6), then it will be referred to as the recovered elastic stress. The relation between the recovered elastic stress and the strain is called the recovered elastic stress–strain relation, which would be the same as the instantaneous elastic stress–strain relation (8) as long as the assumption (5) is strictly valid. From Eqs. (6) and (8), we have

$$\sigma^{\mathrm{e}}(t) = \varphi[\varepsilon(t)] = \int_{-\infty}^{t} D_\sigma^{\mathrm{r}}(t-\tau)\,\mathrm{d}\sigma(\tau). \tag{9}$$

Eq. (9) gives the relation between the current stress $\sigma(t)$ and the prescribed strain $\varepsilon(t)$.

Similarly, suppose that

$$E_i(t-\tau_1, t-\tau_2, \ldots, t-\tau_i) = b_i \prod_{k=1}^{i} E_\varepsilon^{\mathrm{r}}(t-\tau_k), \tag{10}$$

where nondimensional quantity $E_\varepsilon^{\mathrm{r}}(t) = E_\varepsilon(t)/E_\varepsilon$ is referred to as the relative relaxation modulus. The subscript "ε" of $E_\varepsilon^{\mathrm{r}}$ in the ε-relation (11) is used to distinguish E_σ^{r} in the σ-relation (19b). $E_\varepsilon(t)$ is referred to as the relaxation modulus and $E_\varepsilon = E_\varepsilon(0)$ is the instantaneous elastic modulus. Defining

$$\varepsilon^{\mathrm{e}}(t) = E_\varepsilon^{\mathrm{r}} * \mathrm{d}\varepsilon = \int_{-\infty}^{t} E_\varepsilon^{\mathrm{r}}(t-\tau)\,\mathrm{d}\varepsilon(\tau), \tag{11}$$

and following the same procedure as above, we obtain

$$\varepsilon^{\mathrm{e}}(t) = \psi[\sigma(t)], \tag{12}$$

where $\sigma(t)$ is understood as an prescribed input quantity, so that the known stress itself is the instantaneous stress, i.e. $\sigma(t) = \sigma^{\mathrm{e}}(t)$. This corresponds that $D_\sigma^{\mathrm{r}}(t) = H(t)$ in Eq. (6). Eq. (12), which is the inversion of Eq. (8), also defines the instantaneous elastic stress–strain relation. If one calculates the recovered elastic strain $\varepsilon^{\mathrm{e}}(t)$ from the current strain $\varepsilon(t)$ by the hereditary integral (11), then the relation between the recovered elastic strain and the stress is also called the recovered instantaneous elastic relation. From Eqs. (11) and (12), we have

$$\varepsilon^{\mathrm{e}}(t) = \psi[\sigma(t)] = \int_{-\infty}^{t} E_\varepsilon^{\mathrm{r}}(t-\tau)\,\mathrm{d}\varepsilon(\tau). \tag{13}$$

Eq. (13) gives the relation between the current strain $\varepsilon(t)$ and the prescribed stress $\sigma(t)$. Thus, if the current stress response $\sigma(t)$ (or the current strain response $\varepsilon(t)$ is known, from Eq. (6) (or (11)) we can find the recovered elastic stress $\sigma^{\mathrm{e}}(t)$ (or recovered elastic strain $\varepsilon^{\mathrm{e}}(t)$).

Following the treatment of Gurtin and Sternberg (1962), we employ the notation of Stieltjes convolution, and assume that $\sigma(t) = \varepsilon(t) = 0$ for $t < 0$. Because of the axiom of nonretroactivity, $D_\sigma(t) = D_\varepsilon(t) = E_\sigma(t) = E_\varepsilon(t) = 0$ for $t < 0$. Generally, let $\varphi(t)$ and $\psi(t)$ be continuous functions for $0 \leqslant t < \infty$, $\varphi(t) = \psi(t) = 0$ for $t < 0$, the values of the functions at $t = 0$ may have a jump, $\varphi(t) = \varphi(0^+)$, $\psi(t) = \psi(0^+)$ at $t = 0^+$, then the Stieltjes convolution of this two functions can be defined as

$$\varphi * \mathrm{d}\psi = \int_{-\infty}^{t} \varphi(t-\tau)\,\mathrm{d}\psi(\tau) = \psi(0^+)\,\varphi(t) + \int_{0^+}^{t} \varphi(t-\tau)\,\mathrm{d}\psi(\tau), \tag{14}$$

where the first term is the instantaneous one, which reflects the contribution of the jumped value $\psi(0^+)$ at the instant $t = 0^+$. The second term is the hereditary one, which reflects the sum of contributions of the values of $\psi(\tau)$ at instant τ, $0 < \tau < t$, to the value of $\varphi * \mathrm{d}\psi$ at time t. The following properties of commutivity, associativity of the Stieltjes convolution of φ with ψ and ω (also defined over $0 < t < \infty$) and the Stieltjes convolution of a function with the Heaviside unite step function $H(t)$ will be used later:

$$\varphi * \mathrm{d}\psi = \psi * \mathrm{d}\varphi; \tag{15}$$

$$\varphi * \mathrm{d}(\psi * \mathrm{d}\omega) = (\varphi * \mathrm{d}\psi) * \mathrm{d}\omega = \varphi * \mathrm{d}\psi * \mathrm{d}\omega; \tag{16}$$

$$\varphi * \mathrm{d}H = H * \mathrm{d}\varphi = \varphi. \tag{17}$$

We now solve the Volterra integral equation of the second kind (6) (or (11)) for $\sigma(t)$ (or $\varepsilon(t)$): Taking the Stieltjes convolution of E_σ^{r} or $D_\varepsilon^{\mathrm{r}}$ with the functions on both sides of Eq. (6) or Eq. (11), using the properties of Eqs. (15)–(17) and requiring that

$$E_\sigma^{\mathrm{r}} * \mathrm{d}D_\sigma^{\mathrm{r}} = D_\sigma^{\mathrm{r}} * \mathrm{d}E_\sigma^{\mathrm{r}} = H(t), \quad E_\varepsilon^{\mathrm{r}} * \mathrm{d}D_\varepsilon^{\mathrm{r}} = D_\varepsilon^{\mathrm{r}} * \mathrm{d}E_\varepsilon^{\mathrm{r}} = H(t), \tag{18}$$

we may obtain the inverse equations of Eqs. (6) and (11). All the results are summarized as follows:

$$\sigma^{\mathrm{e}}(t) = D_\sigma^{\mathrm{r}} * \mathrm{d}\sigma = \sigma * \mathrm{d}D_\sigma^{\mathrm{r}}, \tag{19a}$$

$$\sigma(t) = E_\sigma^{\mathrm{r}} * \mathrm{d}\sigma^{\mathrm{e}} = \sigma^{\mathrm{e}} * \mathrm{d}E_\sigma^{\mathrm{r}}, \tag{19b}$$

$$\varepsilon^{\mathrm{e}}(t) = E_\varepsilon^{\mathrm{r}} * \mathrm{d}\varepsilon = \varepsilon * \mathrm{d}E_\varepsilon^{\mathrm{r}}, \tag{20a}$$

$$\varepsilon(t) = D_\varepsilon^{\mathrm{r}} * \mathrm{d}\varepsilon^{\mathrm{e}} = \varepsilon^{\mathrm{e}} * \mathrm{d}D_\varepsilon^{\mathrm{r}}. \tag{20b}$$

The instantaneous elastic compliance and the instantaneous elastic modulus have the relation: $E_\sigma = 1/D_\sigma$ (or $E_\varepsilon = 1/D_\varepsilon$). Relations (19a), (19b) and (20a), (20b) are the corresponding relations between the current stress (or strain) and the recovered elastic stress (or strain).

It should be emphasized that in order to describe the σ-relation and the ε-relation uniformly by Eqs. (19a), (19b) and (20a), (20b), we have used different notations $D^{\mathrm{r}}_{\sigma}(t)$ and $D^{\mathrm{r}}_{\varepsilon}(t)$ for the relative creep compliance, they are not necessarily the same quantities. Similarly, $E^{\mathrm{r}}_{\sigma}(t)$ and $E^{\mathrm{r}}_{\varepsilon}(t)$ are not necessarily the same quantity as well. This is due to the approximation of the single integral constitutive equation on the one hand. On the other hand, it is mainly due to the following reason. In the σ–ε relation, one is a prescribed input quantity, its current value is equal to the instantaneous value; and the other is a response quantity, its current value involves hereditary effect. For example, if $\sigma(t)$ is known, then $\sigma(t) = \sigma^{\mathrm{e}}(t)$. From Eqs. (19a), (19b) and (17) we see that in this case, $D^{\mathrm{r}}_{\sigma}(t) = E^{\mathrm{r}}_{\sigma}(t) = H(t)$. Since the response $\varepsilon(t) \neq \varepsilon^{\mathrm{e}}(t)$, thus, $D^{\mathrm{r}}_{\varepsilon}(t) \neq H(t)$, $E^{\mathrm{r}}_{\varepsilon}(t) \neq H(t)$ in Eqs. (20b) and (20a). Similarly, if $\varepsilon(t)$ is known, then $\varepsilon(t) = \varepsilon^{\mathrm{e}}(t)$. From Eqs. (20a), (20b) and (17) we see that in this case, $D^{\mathrm{r}}_{\varepsilon}(t) = E^{\mathrm{r}}_{\varepsilon}(t) = H(t)$. Since the response $\sigma(t) \neq \sigma^{\mathrm{e}}(t)$, thus, $D^{\mathrm{r}}_{\sigma}(t) \neq H(t)$, $E^{\mathrm{r}}_{\sigma}(t) \neq H(t)$ in Eqs. (19a) and (19b). This fact will be used repeatedly in the next section.

Here, the definition of the creep compliance does not depend on the stress, and the definition of relaxation modulus does not depend on the strain. These definitions are different from those in linear viscoelasticity in form, but the meanings are the same. For example, in the case of linear viscoelasticity, if the strain $\varepsilon(t)$ is known, substituting $\sigma^{\mathrm{e}}(t) = E_{\sigma}\varepsilon^{\mathrm{e}}(t) = E_{\sigma}\varepsilon(t)$ into Eq. (19b), we find

$$\sigma(t) = \int_{-\infty}^{t} [E_{\sigma}(t-\tau)/E_{\sigma}]\,\mathrm{d}[E_{\sigma}\varepsilon(\tau)] = \int_{-\infty}^{t} E_{\sigma}(t-\tau)\,\mathrm{d}\varepsilon(\tau). \tag{21}$$

Eq. (21) is the usual constitutive relation in linear viscoelasticity.

2.2. Simplified nonlinear viscoelastic constitutive relations

For a class of nonlinear viscoelastic materials that obey the assumptions (5) and (10), the constitutive relations can be established as follows. We first find the recovered elastic stress and strain from Eqs. (19a) and (20a), and then relate the current stress and current strain in the nonlinear viscoelastic body by using the nonlinear elastic stress–strain relation (1) or (2).

If the strain $\varepsilon(t)$ is known, then $\varepsilon(t) = \varepsilon^{\mathrm{e}}(t)$, which corresponds to $E^{\mathrm{r}}_{\varepsilon}(t) = D^{\mathrm{r}}_{\varepsilon}(t) = H(t)$, from Eqs. (19a), (19b) and (1), we find (see Fig. 1)

$$\partial W/\partial\varepsilon = D^{\mathrm{r}}_{\sigma} * \mathrm{d}\sigma \quad \text{or} \quad \sigma(t) = (\partial W/\partial\varepsilon) * \mathrm{d}E^{\mathrm{r}}_{\sigma}. \tag{22a}$$

In the special case of linear viscoelasticity, $\sigma^{\mathrm{e}}(t) = E_{\sigma}\varepsilon^{\mathrm{e}}(t) = E_{\sigma}\varepsilon(t)$, Eq. (22a) can be reduced to:

$$\sigma(t) = \varepsilon * \mathrm{d}E_{\sigma}. \tag{22b}$$

Similarly, if the stress $\sigma(t)$ is known, then $\sigma(t) = \sigma^{\mathrm{e}}(t)$, which corresponds to $D^{\mathrm{r}}_{\sigma}(t) = E^{\mathrm{r}}_{\sigma}(t) = H(t)$, from Eqs. (20a), (20b) and (2), we find (see Fig. 2)

$$\partial W_{\mathrm{c}}/\partial\sigma = E^{\mathrm{r}}_{\varepsilon} * \mathrm{d}\varepsilon \quad \text{or} \quad \varepsilon(t) = (\partial W_{\mathrm{c}}/\partial\sigma) * \mathrm{d}D^{\mathrm{r}}_{\varepsilon}. \tag{23a}$$

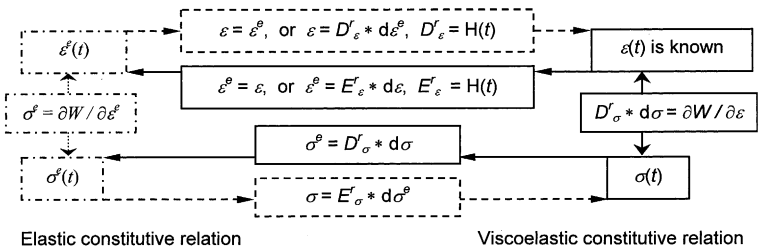

Fig. 1. If the strain ε is known, the viscoelastic constitutive relation can be established by Eqs. (19a), (19b), (20a) and (20b) and the elastic constitutive relation (1).

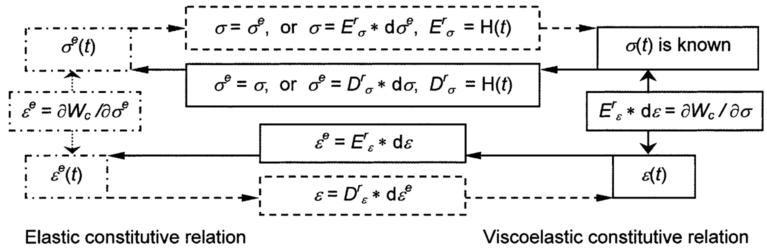

Fig. 2. If the stress σ is known, the viscoelastic constitutive relation can be established by Eqs. (19a), (19b), (20a) and (20b) and the elastic constitutive relation (2).

In the special case of linear viscoelasticity, $\varepsilon^e(t) = D_\varepsilon \sigma^e(t) = D_\varepsilon \sigma(t)$, Eq. (23a) can be reduced to:

$$\varepsilon(t) = \sigma * dD_\varepsilon. \tag{23b}$$

It is seen that Eq. (22b) (or (23b)) is the same as the constitutive equation of the relaxation type (or creep type) in linear viscoelasticity.

3. Elasticity recovery correspondence principles

Utilizing the similarity of the constitutive relations between the elastic body and the viscoelastic body, two kinds of approaches for deducing the correspondence principle in linear viscoelasticity have been developed: the integral transform method and the Volterra's principle. It is rather difficult to generalize them to the case of nonlinear viscoelastic problems. Thus, an entirely different approach will be employed. Not the similarity between constitutive relations will be utilized, but the recoverability from the nonlinear viscoelastic current stress (or strain) to the instantaneous elastic stress (or strain), Eqs. (19a) and (20a), will be utilized. The current stress, strain and displacement obey the laws of nonlinear viscoelasticity, whereas the recovered instantaneous elastic stress, strain and displacement obey the laws of nonlinear elasticity. We will now establish the corresponding relations between the solutions of the nonlinear viscoelastic problem and the nonlinear elastic problem. The following correspondence principles are equally applicable to the special case of linear viscoelastic problems. Their validity can also be verified by the known linear viscoelastic solutions.

3.1. Governing equations for nonlinear elastic problem in one-dimensional case

Consider a one-dimensional body with surface boundary $S = S_U \cup S_T$ at the two ends (Fig. 3). Let the body force per unit volume in the x-direction be prescribed by F^e in the body, and let the displacement be prescribed over S_U by U^e and the traction be prescribed over S_T by T^e. Then governing equations for the quasi-static, isotropic, nonlinear elastic problem in one-dimensional case are as follows:

Equilibrium equation $\quad (\partial \sigma^e / \partial x) + F^e = 0, \qquad (24)$

Geometrical equation $\quad \varepsilon^e = \partial u^e / \partial x, \qquad (25)$

Constitutive equation $\quad \varepsilon^e = \partial W_c / \partial \sigma^e \qquad (26)$

or $\quad \sigma^e = \partial W / \partial \varepsilon^e, \qquad (27)$

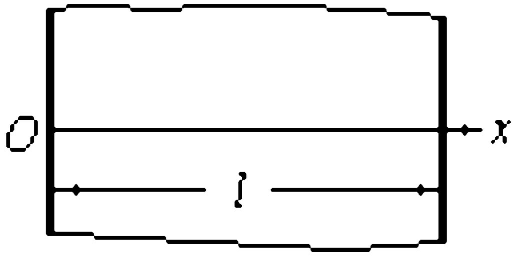

Fig. 3. One-dimensional body.

Boundary conditions of the first kind $\begin{cases} \sigma^{e} = T^{e}, & \text{on } S_T, \\ u^{e} = U^{e} = 0, & \text{on } S_U, \end{cases}$ (28)

Boundary condition of the second kind $u^{e} = U^{e}, \quad \text{on } S_U = S.$ (29)

3.2. Governing equations for nonlinear viscoelastic problem in one-dimensional case

Assume that the geometry and the boundary conditions are the same as those for the corresponding nonlinear elastic body: the body force per unit volume in the x-direction is prescribed by $F = F^{e}$ in the body; and $T(t) = T^{e}(t)$ (or $U = U^{e}$) is prescribed over the boundary. Then the governing equations for quasi-static, isotropic, nonlinear viscoelastic problem in one-dimensional case are as follows:

Equilibrium equation $(\partial\sigma/\partial x) + F = 0,$ (30)

Geometrical equation $\varepsilon = \partial u/\partial x,$ (31)

Constitutive equation $\varepsilon(t) = (\partial W_c/\partial\sigma) * \mathrm{d}D^{r}_{\varepsilon}$ (32)

or $\sigma(t) = (\partial W/\partial\varepsilon) * \mathrm{d}E^{r}_{\sigma}.$ (33)

Boundary conditions of the first kind $\begin{cases} \sigma = T, & \text{on } S_T, \\ u = U = 0, & \text{on } S_U, \end{cases}$ (34)

Boundary condition of the second kind $u = U, \quad \text{on } S_U = S.$ (35)

If the body is originally undisturbed, the initial conditions take the form

$$\sigma(t) = \varepsilon(t) = u(t) = 0, \quad \text{for } -\infty < t < 0.$$

3.3. Elasticity recovery correspondence principles

Using Eqs. (20a), (20b), (25) and (31) and the interchangeability of the differentiation with respect to the spatial variable x and the hereditary integration with respect to the variable t, it is easy to obtain the relations between the current displacement and the recovered elastic displacement:

$$u^{e}(t) = E^{r}_{\varepsilon}(t) * \mathrm{d}u(t), \quad u(t) = D^{r}_{\varepsilon}(t) * \mathrm{d}u^{e}(t). \tag{36}$$

Similarly, from Eqs. (19a), (19b), first terms of Eqs. (28), (34) and (36), (29), (35) we have, respectively,

$$T^{\mathrm{e}}(t) = D^{\mathrm{r}}_{\sigma} * \mathrm{d}T, \quad T(t) = E^{\mathrm{r}}_{\sigma} * \mathrm{d}T^{\mathrm{e}}. \tag{37}$$

$$U^{\mathrm{e}}(t) = E^{\mathrm{r}}_{\varepsilon} * \mathrm{d}U, \quad U(t) = D^{\mathrm{r}}_{\varepsilon} * \mathrm{d}U^{\mathrm{e}}. \tag{38}$$

We are now ready to give the elasticity recovery correspondence principles (ERCPs).

ERCP-1: If the body is originally undisturbed, the body force per unit volume is prescribed by F in the body and the boundary conditions of the first kind are prescribed over the surface, then the solution of the nonlinear viscoelastic problem (i.e., Eqs. (30)–(32) and (34)) is as follows:

$$\sigma(t) = \sigma^{\mathrm{e}}(t), \quad \varepsilon(t) = D^{\mathrm{r}}_{\varepsilon} * \mathrm{d}\varepsilon^{\mathrm{e}}, \quad u(t) = D^{\mathrm{r}}_{\varepsilon} * \mathrm{d}u^{\mathrm{e}}, \tag{39}$$

where the Stieltjes convolution is defined by Eq. (14), and ε^{e}, σ^{e}, u^{e} satisfy equations of the corresponding nonlinear elastic problem (i.e., Eqs. (24)–(26) and (28)) together with the same boundary conditions: $F^{\mathrm{e}} = F$ in the body, $T^{\mathrm{e}} = T$ over S_T and $U^{\mathrm{e}} = U = 0$ over S_U.

Proof: Since the body force and surface force are prescribed for this kind of problems, they are the instantaneous values: $F = F^{\mathrm{e}}$; $T = T^{\mathrm{e}}$, it follows from Eqs. (37) and (17) that $D^{\mathrm{r}}_{\sigma}(t) = E^{\mathrm{r}}_{\sigma}(t) = H(t)$. Substituting them into Eqs. (19a) and (19b) and using Eq. (17), we obtain the first term of Eq. (39). In this case, the stress in nonlinear viscoelastic body coincides with the stress in the corresponding nonlinear elastic body. Since the prescribed surface displacement $U^{\mathrm{e}} = U = 0$, then Eq. (38) has no restriction on $D^{\mathrm{r}}_{\varepsilon}(t)$ and $E^{\mathrm{r}}_{\varepsilon}(t)$.

That Eq. (39) is indeed the solution is easily proved by substituting it into the corresponding equations. Substituting $\sigma = \sigma^{\mathrm{e}}$ into Eqs. (30), (32) and the first term of Eq. (34), using Eq. (20b), and noting that $F = F^{\mathrm{e}}$; $T = T^{\mathrm{e}}$, we obtain the same equations as Eqs. (24), (26) and the first term of Eq. (28), which are satisfied naturally. Substituting the last two terms of Eq. (39) into Eq. (31) and the second term of Eq. (34), and noting that $U = U^{\mathrm{e}}$, we obtain

$$D^{\mathrm{r}}_{\varepsilon} * \mathrm{d}\varepsilon^{\mathrm{e}} = \partial(D^{\mathrm{r}}_{\varepsilon} * \mathrm{d}u^{\mathrm{e}})/\partial x = D^{\mathrm{r}}_{\varepsilon} * \mathrm{d}(\partial u^{\mathrm{e}}/\partial x), \quad D^{\mathrm{r}}_{\varepsilon} * \mathrm{d}u^{\mathrm{e}} = 0.$$

Here we have used the interchangeability of the differentiation with respect to the spatial variable and the hereditary integration with respect to time. Due to Eq. (25) and the second term of Eq. (28), the above equations are also satisfied. By the definition (14), the initial condition is automatically satisfied. Therefore, Eq. (39) is indeed the solution of the nonlinear viscoelastic problem.

ERCP-2: If the body is originally undisturbed, there are no body forces ($F = 0$) in the body and boundary condition of the second kind is prescribed over the surface, then the solution of the nonlinear viscoelastic problem (i.e., Eqs. (30), (31), (33) and (35)) is as follows:

$$u(t) = u^{\mathrm{e}}(t), \quad \varepsilon(t) = \varepsilon^{\mathrm{e}}(t), \quad \sigma(t) = E^{\mathrm{r}}_{\sigma} * \mathrm{d}\sigma^{\mathrm{e}}, \tag{40}$$

where the Stieltjes convolution is defined by Eq. (14), and ε^{e}, σ^{e}, u^{e} satisfy the equations of the corresponding nonlinear elastic problem (i.e., Eqs. (24), (25), (27) and (29)) together with the same boundary conditions: $F^{\mathrm{e}} = F = 0$ in the body, and $U^{\mathrm{e}} = U$, on the boundary.

Proof: Since the displacement is prescribed for this kind of problems, it is the instantaneous value: $U = U^{\mathrm{e}}$, it follows from Eqs. (38) and (17) that $E^{\mathrm{r}}_{\varepsilon}(t) = D^{\mathrm{r}}_{\varepsilon}(t) = H(t)$. Substituting them into Eqs. (36), (20a) and (20b) and using Eq. (17), we obtain the first two terms of Eq. (40). In this case, the displacement and strain in nonlinear viscoelastic body coincide with the displacement and strain in the corresponding nonlinear elastic body.

That Eq. (40) is indeed the solution is easily proved by substituting it into the corresponding equations. Substituting the first two terms of Eq. (40) into Eqs. (31) and (35) and noting that $U_i = U^{\mathrm{e}}_i$, we obtain the same equations as Eqs. (25), and (29), which are satisfied naturally. Substituting the last term of Eq. (40) into Eqs. (30) and (33), noting that $F = F^{\mathrm{e}} = 0$, and using the second term of Eq. (40), we obtain

$$\partial(E^{\mathrm{r}}_{\sigma} * \mathrm{d}\sigma^{\mathrm{e}})/\partial x = E^{\mathrm{r}}_{\sigma} * \mathrm{d}(\partial\sigma^{\mathrm{e}}/\partial x) = 0,$$
$$E^{\mathrm{r}}_{\sigma} * \mathrm{d}\sigma^{\mathrm{e}} = \sigma^{\mathrm{e}} * \mathrm{d}E^{\mathrm{r}}_{\sigma} = (\partial W/\partial\varepsilon^{\mathrm{e}}) * \mathrm{d}E^{\mathrm{r}}_{\sigma}.$$

Due to Eqs. (24) and (27), the above equations are also satisfied. The initial condition is automatically satisfied as before. Therefore, Eq. (40) is indeed the solution of the nonlinear viscoelastic problem.

4. Comparison between theory and experiments

To verify the above theory, the experiment of quasi-static equal-amplitude strain history in uniaxial stress state has been done, which was performed on specimens of modified polypropylene. The specimen size was $250 \times 50 \times 3.8$ mm^3, with cross-section area $A = 192.66$ mm^2 and gauge length 50 mm. A nine cyclic equal-amplitude-strain loading and unloading test was performed on a material test machine INSTRON with strain rate of ± 0.00127 s^{-1} and strain amplitude of 0.029. Experimental data were inputted to a computer. The stress was calculated by dividing the force by initial area and the corresponding strain was calculated by dividing the relative displacement between the two ends of the gauge length by the initial gauge length. Experimental results, including the time t, the prescribed strain $\varepsilon(t)$ and the current stress $\sigma(t)$, were processed by Origin software and numerical calculations were performed by MathCAD software. The prescribed ε–t curve is shown as the solid line in Fig. 4. The experimental points of the σ–t relation and the σ–ε relation are shown as '$\Diamond$' in Figs. 5 and 6, respectively.

As there are no stresses acting on all sections that parallel to x-axis in the one-dimensional case, the solution of the problem under consideration is the same as the corresponding one-dimensional solution. In the problem under consideration, the body force is zero and the displacement u is prescribed over the two ends of the specimen. According to the ERCP-2, the solution of the problem is determined by Eq. (40)

$$\varepsilon(t) = \varepsilon^{\mathrm{e}}(t), \quad \sigma(t) = E^{\mathrm{r}}_{\sigma} * \mathrm{d}\sigma^{\mathrm{e}} = \sigma^{\mathrm{e}} * \mathrm{d}E^{\mathrm{r}}_{\sigma}.$$

Substituting the instantaneous stress–strain relation (8) $\sigma^{\mathrm{e}} = \varphi[\varepsilon(t)]$, which will be found in the following, into the last term of Eq. (40), we can predict the current stress as follows:

$$\sigma(t) = \sigma^{\mathrm{e}} * \mathrm{d}E^{\mathrm{r}}_{\sigma} = \varphi[\varepsilon(t)] * \mathrm{d}E^{\mathrm{r}}_{\sigma}(t) = \varphi[\varepsilon(t)] + \int_{0^+}^{t} \varphi[\varepsilon(t-\tau)] * \mathrm{d}E^{\mathrm{r}}_{\sigma}(\tau). \tag{41}$$

We may use the following analytic expressions, which can fit the experimental data very well, for the relative creep compliance D^{r}_{σ} and the relative relaxation modulus E^{r}_{σ} (Zhang, 1999):

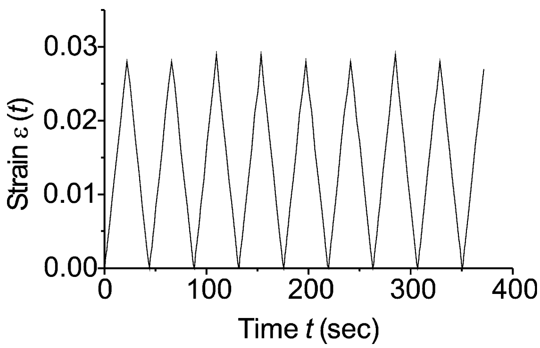

Fig. 4. Prescribed uniaxial equal-amplitude-strain loading and unloading curve.

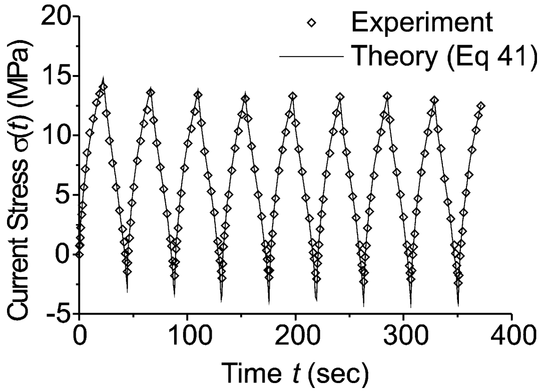

Fig. 5. Current stress–time curve.

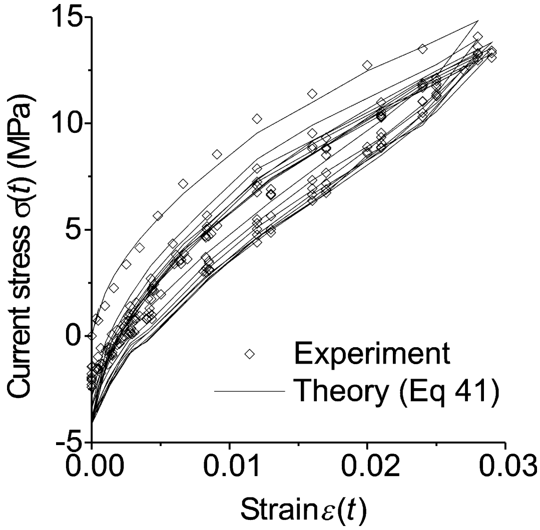

Fig. 6. Current stress–strain curve.

$$D_\sigma^r(t) = D_\sigma(t)/D_\sigma = 1 + [(D_{\sigma\infty}/D_\sigma) - 1][1 - \exp(-\beta[(1+\alpha)t]^{1-\alpha})], \tag{42}$$

$$E_\sigma^r(t) = E_\sigma(t)/E_\sigma = 1 - [1 - (E_{\sigma\infty}/E_\sigma)][1 - \exp(-\beta_1[(1+\alpha)t]^{1-\alpha})], \tag{43}$$

where the parameters α, β, β_1 ($0 < \alpha < 1$, $\beta > 0$, $\beta_1 > 0$), instantaneous compliance D_σ, long-time compliance $D_{\sigma\infty}$, instantaneous modulus E_σ and long-time modulus $E_{\sigma\infty}$ are determined by experiments. Usually, the instantaneous elastic stress–strain relation $\sigma^e = \varphi(\varepsilon)$ in Eq. (41) can be obtained from the isochronous curve at $t = 0$, which is determined from a series of creep curves (or relaxation curves) under different stresses (or strains) (Rabotnov, 1980; Sun, 1999). Here a simplified method for finding $\sigma^e = \varphi(\varepsilon)$ will be employed as follows. Substituting the experimental data of the current stress $\sigma(t)$ into Eq. (6), we may calculate the recovered elastic stress $\sigma^e(t)$. If the property of the material satisfies the assumption (5) strictly, the experimental points of the recovered elastic stress–strain relation σ^e–ε in the nine loading–

unloading cycles must fall in a single curve. In other words, Eq. (6) will shift all the experimental points '◇' in Fig. 6 upward into a single curve, which is the same as the instantaneous elastic stress–strain curve that we want to find. However, for real materials, because of the approximation of the Eq. (6), usually, these recovered elastic stress–strain points ('◇' in Fig. 7) can only approximately fall in the instantaneous elastic stress–strain curve. Now we may adjust the values of α, β, $D_{\sigma\infty}/D_\sigma$ in Eq. (42) to fit the points ('◇' in Fig. 7) in nine loading–unloading cycles into a single curve as close as possible, and then use the intermediate values of the recovered elastic stress–strain points as the instantaneous stress–strain relation. We use the following analytic expression for the instantaneous elastic stress–strain curve to fit these points:

$$\sigma^e = \varphi(\varepsilon) = k\varepsilon\{1 - c[1 - \exp(-b[(1+a)\varepsilon]^{(1-a)})]\}, \tag{44}$$

where $k = 4300$ MPa, $a = 0.63$, $b = 8.8$, $c = 0.88$. The instantaneous elastic stress–strain curve calculated by Eq. (44) is shown by the solid lines in Figs. 7 and 8. Finally, we adjust the value of β_1 to satisfy Eq. (18). The parameters thus determined are: $\alpha = 0.67$, $\beta = 0.199\ \mathrm{s}^{\alpha-1}$, $\beta_1 = 0.35\ \mathrm{s}^{\alpha-1}$, $E_{\sigma\infty}/E_\sigma = D_\sigma/D_{\sigma\infty} = 0.44$.

With the instantaneous elastic stress–strain relation $\sigma^e(t) = \varphi[\varepsilon(t)]$ and the necessary material kernel functions $E^r_\sigma(t)$ and $D^r_\sigma(t)$, the current stress $\sigma(t)$ can be predicted by substituting Eq. (44) into Eq. (41). The theoretical current stress–time curve is shown by the solid line in Fig. 5, and the theoretical current stress–strain curve is shown by the solid line in Fig. 6. The agreement between the experiment ('◇' in Figs. 5 and 6) and the theory (solid line in Figs. 5 and 6) is considerably good in all the nine cycles. This shows the practicability of the constitutive relations of the single integral form, (32) and (33), and the validity of the elasticity recovery correspondence principles in one-dimensional case.

Recovered elastic stress is determined by Eq. (6)

$$\sigma^e(t) = D^r_\sigma * \mathrm{d}\sigma = D^r_\sigma(t)\sigma(0) + \int_{0^+}^{t} D^r_\sigma(t-\tau)\,\mathrm{d}\sigma(\tau). \tag{45}$$

Substituting the theoretical values of the current stress (41) into Eq. (45), we obtain the recovered elastic stress:

$$\sigma^e(t) = D^r_\sigma * \mathrm{d}\sigma = D^r_\sigma * \mathrm{d}[\varphi(\varepsilon) * \mathrm{d}E^r_\sigma] = \varphi[\varepsilon(t)]. \tag{46}$$

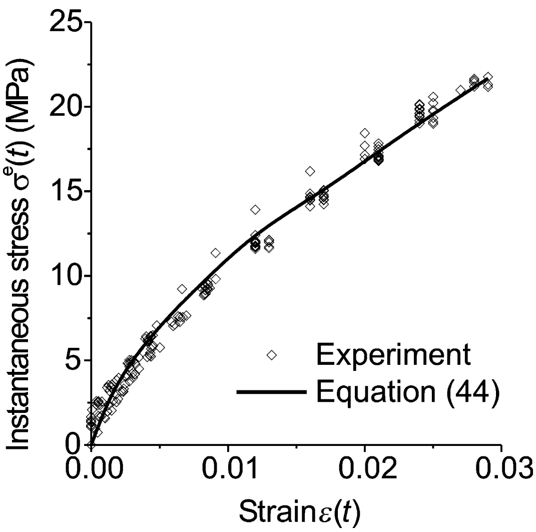

Fig. 7. Instantaneous stress–strain curve.

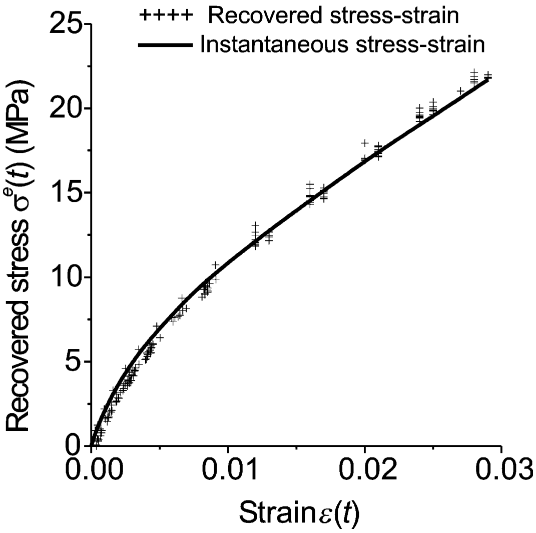

Fig. 8. Recovered elastic stress–strain curve.

Eq. (46) shows that the recovered stress–strain curve coincides with the instantaneous elastic curve exactly. Recovered elastic stress–strain relation calculated from the theoretical values of $\sigma(t)$ in the nine cycles by Eq. (45) is shown by the symbols '+' in Fig. 8. It is shown from this figure that the nine loading and unloading curves coincide with the instantaneous elastic stress–strain curve (the solid line in Fig. 8), and the recovered elastic stress and strain return to the origin simultaneously, which satisfies the requirements of elastic behavior completely. This shows the validity of the correspondence between the nonlinear elastic and nonlinear viscoelastic constitutive relations. A small difference in the numerical results between the recovered stress–strain curve and the instantaneous stress–strain curve is due to that Eqs. (42) and (43) can only approximately satisfy Eq. (18) when we adjust the parameter β_1 by means of the numerical technique, and the maximum error is about 3%.

It is interesting to make a simple comparison between the work of Rabotnov, Schapery and ours. Apart from the slight difference of the notations, the left-hand side of Eq. (8) was called modified stress by Rabotnov and pseudostress by Schapery. The main difference between the work of Rabotnov, Schapery and ours is as follows. In our work, the prescribed values for the corresponding instantaneous elastic problem are the same as those for the viscoelastic problem: $F^e = F$, $T^e = T$, $U^e = U$. What we have established is the correspondence between the solutions of the nonlinear viscoelastic body and the nonlinear instantaneous elastic body. Whereas in the work of Rabotnov and Schapery, the prescribed values for the corresponding pseudo- (or modified) problem are calculated from those for the viscoelastic problem by taking hereditary integrals: $F^R = E_{R1}(D_1 * dF)$, $T^R = E_{R1}(D_1 * dT)$, $U^R = E_R^{-1}(E * dU)$, where F^R, T^R, U^R and E_R are the prescribed body force per unit volume, the prescribed traction, the prescribed displacement and the modulus in the reference (or pseudo) elastic problem, respectively. The subscript 1 is used to the indicate that quantities E_{R1} and E_R (as well as the creep compliances D_1 and D) are not necessarily the same. What Schapery had established was the correspondence relation between the solutions of the nonlinear viscoelastic body and the nonlinear pseudobody. However, pseudostrain ε^R and pseudostress σ^R are not in general the physical variables and the pseudobody is only an elastic-like body as will be seen later.

Now we use Schapery's correspondence principle to predict the current stress when the experimental strain history is prescribed for a comparison. According to the CP-1 (Schapery, 1984), the solution of the above problem is

$$\sigma(t) = E_{\mathrm{R1}}^{-1}(E_1 * \mathrm{d}\sigma^{\mathrm{R}}), \quad u = E_{\mathrm{R}}(D * \mathrm{d}u^{\mathrm{R}}), \tag{47}$$

where σ^{R} and u^{R} satisfy the equations of the pseudoelastic problem, together with the boundary condition $u^R = U^{\mathrm{R}} = E_{\mathrm{R}}^{-1}(E * \mathrm{d}U)$ at the two ends. The solution of the pseudoelastic problem is

$$\varepsilon^{\mathrm{R}}(t) = E_{\mathrm{R}}^{-1} \int_0^t E(t-\tau)\,\mathrm{d}\varepsilon(\tau). \tag{48}$$

When the strain is prescribed, Schapery used the current stress–pseudostrain relation $\sigma(t)$–$\varepsilon^{\mathrm{R}}(t)$ to define the pseudoelastic constitutive relation (Schapery, 1982). It follows that $\sigma(t) = \sigma^{\mathrm{R}}(t)$, or $E_{\mathrm{R1}}^{-1}E_1(t) = H(t)$. To predict the current stress, we must find the pseudoelastic constitutive relation $\sigma^{\mathrm{R}} = \partial W/\partial\varepsilon^{\mathrm{R}}$ and the material kernels $E_{\mathrm{R}}^{-1}E(t)$ at first. To this end, we use the same expression (43) for the analytic expression of $E_{\mathrm{R}}^{-1}E(t)$. Substituting the strain history $\varepsilon(t)$ into Eq. (48) and using the experimental data $\sigma(t)$, we may obtain the experimental points in the $\sigma(t)$–$\varepsilon^{\mathrm{R}}(t)$ plot ('◇' in Fig. 9). These points are obtained by Eq. (48), which shifts all the experimental points in the $\sigma(t)$–$\varepsilon(t)$ plot (see Fig. 10) leftward. We now adjust the parameters α, β, β_1, $E_{\mathrm{R}\infty}/E_{\mathrm{R}}$ to fit the points ('◇' in Fig. 9) in nine loading–unloading cycles into a single curve as close as possible. The parameters thus determined are: $\alpha = 0.76$, $\beta = 0.109\,\mathrm{s}^{\alpha-1}$, $\beta_1 = 0.2\,\mathrm{s}^{\alpha-1}$, $E_{\sigma\infty}/E_\sigma = D_\sigma/D_{\sigma\infty} = 0.445$. Then, we use the intermediate values for the $\sigma^{\mathrm{R}}(t)$–$\varepsilon^{\mathrm{R}}(t)$ relation and use the following analytic expression to fit these values:

$$\sigma^{\mathrm{R}}(t) = 900\left|\varepsilon^{\mathrm{R}}\right|^{0.97}H(\varepsilon^{\mathrm{R}}) - 3000\left|\varepsilon^{\mathrm{R}}\right|^{1.53}H(\varepsilon^{\mathrm{R}}) - 900\left|\varepsilon^{\mathrm{R}}\right|^{0.97}H(-\varepsilon^{\mathrm{R}}). \tag{49}$$

The pseudostress–pseudostrain curve calculated by Eq. (49) is shown by the solid line in Fig. 9. As we can see from Fig. 9 that in the nine loading and unloading cycles, when $\varepsilon(t)$ returns to zero, the values of $\sigma^{\mathrm{R}}(t)$ and $\varepsilon^{\mathrm{R}}(t)$ do not return to zero, but have negative values. Such behavior of the pseudobody was called elastic-like behavior by Schapery. Substituting the values of Eq. (49) into the first term of Eq. (47), we obtain the theoretical current stress. If $E_{\mathrm{R1}}^{-1}E_1(t) = H(t)$, then $\sigma(t) = \sigma^{\mathrm{R}}(t)$. The theoretical stress–strain curve, $\sigma(t)$–$\varepsilon(t)$, is shown by the solid line in Fig. 10, and the experimental data is shown by the symbols '◇'.

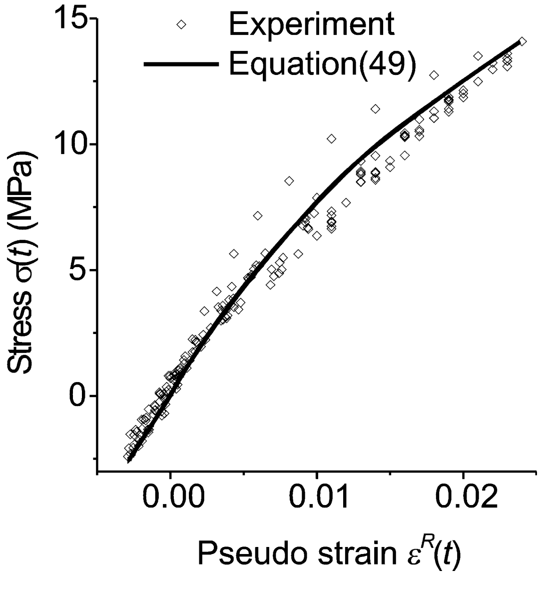

Fig. 9. Stress–pseudostrain curve.

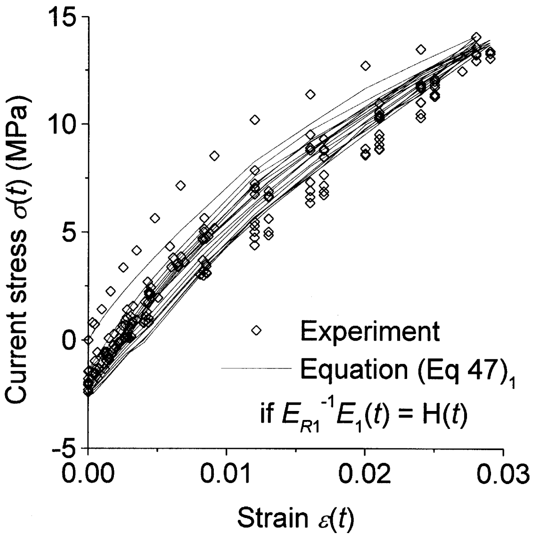

Fig. 10. Current stress–strain curve.

The accuracy of the theory in Fig. 10 (Schapery's method) is much less then that in Fig. 6 (our method). Moreover, in Schapery's CP-1 (Schapery, 1984), it is not clear how to determine the material kernels in different boundary conditions. If $E_{\mathrm{R1}}^{-1}E_1(t) = E_{\mathrm{R}}^{-1}E(t) \neq H(t)$, then $\sigma(t) \neq \sigma^{\mathrm{R}}(t)$. Using the same method as above, we find that the $\sigma^{\mathrm{R}}(t)$–$\varepsilon^{\mathrm{R}}(t)$ curve cannot pass through the origin.

5. Concluding remarks

Nonlinear viscoelastic constitutive relation in the form of simplified single integral is adopted in this paper. Based on this simplified single integral, relations between current stress (or strain) and recovered elastic stress (or strain) are given and two elasticity recovery correspondence principles for solving the nonlinear viscoelastic problems in one-dimensional case are proposed. From the comparison between the theory and the experiment for modified polypropylene, the practicability of the constitutive relations (32) and (33) and the validity of the elasticity recovery correspondence principles are verified for such a class of materials in one-dimensional case. The validity of elasticity recovery correspondence principles can also be proved by the special case of the known linear viscoelastic solutions. In principle, the above theory can be applied whenever the assumptions (5) and (10) are satisfied, or whenever the effects of the strains (or stresses) at different past time upon the present stress (or strain) have negligible interference with each other. At the first glance, these assumptions seem very special; nevertheless for some polymers, some aluminum alloys, some alloy steels at high temperature and some soft soils, the practicability of the simplified single integral constitutive relation based upon these assumptions was verified by experiments (Rabotnov, 1980; Sun, 1999).

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